توزيع باريتو الأسي المركب لأوقات الحياة الجديدة
الخصائص والتقدير

الباحثة : آيات فاخر طاهر هليل   أ.م.د بهاء عبد الرزاق قاسم

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المستخلص:

يهدف هذا البحث إلى اقتراح توزيع احتمالي مركب مقترح يدعى توزيع باريتو–الأسي بأستعمال طريقة العائلة الأسية الجديدة . كما تم اشتقاق الدالة الرياضية للكثافة الاحتمالية ودالة التوزيع التراكمية للتوزيع الجديد (NLTE–Pa) ، بالإضافة إلى خصائصه المتمثلة في (العزم اللامركزي ، العزم المركزي ، معامل الالتواء ، معامل التفلطح ، معامل الأختلاف ، والوسيط). فضلاً عن مناقشة الطرق المختلفة لتقدير المعلمات ، ممثلة بـ (طريقة الامكان الأعظم ، طريقة تقدير أندرسون دارلنك ، وطريقة المربعات الصغرى) من خلال أسلوب المحاكاة بالاستعانة برنامج Mathematica ، والذي توصل الى أفضلية طريقة الأمكان الأعظم .

الكلمات المفتاح : توزيع باريتو ، العائلة الأسية ، الخصائص ، التقدير، أسلوب المحاكاة .

The New Lifetime Exponential - Pareto Distribution: properties and Estimation

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Abstract:

This research aims to suggest a proposal of compound probability distribution which is called the Pareto-exponential distribution using the new exponential family method. The mathematical function of the probability density and the cumulative distribution function of the new distribution (NLTE-Pa) were also derived, in addition to its properties such as (Non-centric moment, central moment, Coefficient of Skewness, Coefficient of Kurtosis, coefficient of variation and median) . As well as discussing the different methods for estimating the parameters, represented by (the method of estimation Maximum Likelihood, the method of estimation by Anderson Darling and the method of least squares) through the simulation method using the Mathematica program, which concluded the preference of the method of the Maximum Likelihood .

Keywords: Pareto distribution, exponential family, Properties, estimation, Simulation.

INTRODUCTION

Statistical distributions are used more widely in the study of many phenomena, and over the years many probability distributions have been built and studied, however, they have a limited range of capabilities and therefore cannot be used in all situations, and this is due to the fact that the characteristics of the phenomenon change over time, and as a result The statistical researcher faces multiple problems and difficulties during the process of statistical analysis, and at the forefront of these difficulties is determining the appropriate probability distribution for the data of the studied phenomenon as a result of the complexity in the behavior of the real data of the phenomena in recent times represented by the high degree of skewness or flattening or the limitation of the appearance of data values in the right side without the left And vice versa. Which made limiting the modeling of phenomena to the single traditional probability distributions of both continuous and intermittent types is not feasible, then it required the construction of new probability distributions that are more flexible in describing the behavior of the real data of the phenomena and adaptable to improve the fit of the data. There is a set of statistical methods and methods used in building the most appropriate distributions, including the method of cutting off probability distributions and the method of installing probability distributions (which is one of the methods of merging probability distributions).

The research problem is determined by the difficulty of adapting traditional probability distributions to represent the data of many phenomena at the present time as a result of the technological development that has made these phenomena' data characterized by high fluctuation between ups and downs. It does not adopt a specific style.

This prompted many researchers to build new probabilistic models to fit the data of these phenomena.

The research aims to propose a model for a proposed probability distribution (Exponential -Pareto type1) that belongs to the(exponential – X family) presented by (Huo et al.) in (2020), as the Pareto distribution of the first kind is used. type 1) as a basic distribution to obtain the proposed distribution represented by the Pareto-exponential distribution (The New Lifetime exponential-Pareto family) (NLTE-Pareto), as well as the derivation of the probability density function and the cumulative distribution function, and the derivation of the structural characteristics of the proposed distribution (Non-centric moment, central moment, Coefficient of Skewness, Coefficient of Kurtosis, coefficient of variation and median) .

There have been many studies and researches dealing with the structure of probability distributions, namely:

In 2012, **(Zea and Silva et al.)** studied the (Pareto exponential - beta) distribution, and some statistical properties were derived (central torque, quantitative function, moment generating function, decentralized torque, average deviations), and estimated Its parameters using the method of greatest possibility, the information matrix (Fisher) was also derived, as well as clarifying the flexibility of the new distribution by applying it to a sample of real data related to exposure to bladder cancer. In 2019, **(Mjely)** studied a compound probability distribution (Weibull-Rayleigh) with three parameters, as some structural characteristics of this distribution were studied such as (collective function, probability density function, survival function, and risk function), as well as estimating the distribution parameters The composite and the survival function using three methods represented by (the method of greatest place, the method of ordinary least squares, the method of weighted least squares (WLS)), and the comparison between the methods of estimation using the statistical standard (MSE), also the composite distribution was compared with some traditional distributions using a number of criteria The trade-off is (Akaiki information criterion, Corrected Akaiki information criterion, Bayesian information criterion) to determine the preference of the composite distribution to represent real data of breast cancer patients. In the year 2020, **(Huo et al.)** presented a sub-probability model of the complex distributions, namely (Weibull-exponential) and (Rayleigh-exponential) distribution, using the new exponential family (NLTE-X Family), and the complex distribution (Weibull-exponential) was studied) in detail and derived some of its properties, as well as estimating its three parameters using the method of maximum possibility (MLE), and it was applied to real data. • In the same year 2021, **(AImongy and Almetwally et al.)** proposed a new compound distribution using the family (EOW-G Family) called the (Rayleigh -Weibull) distribution, and some of its structural properties (simple linear representation, quantitative function, median, generated function) were derived , and torque), as well as estimating its parameters in three ways (maximum potential, maximum spacing, and Bayesian estimation methods). COVID-19 deaths in Italy, Mexico and the Netherlands . In the year 2022, **(Abdul Hassan)** built a new probabilistic model called (NLTE-PF) using the new exponential family, and some structural characteristics of by (Non-centric moment, central moment, Coefficient of Skewness, Coefficient of Kurtosis, coefficient of variation, moment-generating function, and characteristic function), in addition to estimating the parameters and reliability function of the proposed model using four methods (the method of estimation Maximum Likelihood, least squares method, partial estimators, maximum method The result of the divergence estimators), a simulation study was conducted using the (Monte Carlo) method to compare the methods of estimating the parameters and the reliability function of the proposed probabilistic model based on the statistical standard of mean squares integral error. In the year 2022, **(Sahib)** studied the complex distribution called the (exponential-Frechet) distribution with three parameters. Three methods for estimating the distribution parameters (the method of greatest possibility, the method of weighted least squares, the method of Anderson Darling), as well as the use of the statistical scale (mean square error) in order to compare the estimation methods, and the study was conducted on a real sample of patients with renal failure.

This research differs from the previous researches and studies in that a probability distribution was built based on the definition of the family of exponential life times and the derivation of each of the functions of density, distribution, reliability and risk as well as the derivation of its characteristics and the estimation of its parameters by a set of methods.

The New Lifetime exponential Family (NLTE-X Family)

The researcher (Huo Xiaoyan et al.) in (2020) presented a new method for constructing new probability distributions called the New Lifetime exponential-X family, which is one of the families that were generated through the method of fitting distributions (T-X family). As introduced by Alzaatreh in 2013, the new exponential family is characterized by the fact that it makes the distributions more flexible in data representation and makes the risk function in the form of a bathtub, which gives the resulting distributions great importance in data modeling. The cumulative function of the new exponential family is defined by equation (1) below:

$$G(x,\theta,£)_{\text{nlte-pa}} = 1-\left(\frac{1-F(x,£)}{e^{\theta F(x,£)}}\right) \qquad \theta>0\ ;\ x\in R \qquad \dots (1)$$

As for the probability density function of the new exponential family, it is defined by the following equation (2):

$$g(x,\theta,£)_{\text{nlte-pa}} = \frac{f(x,£)}{e^{\theta\, F(x,£)}}\left[1+\theta\bar{F}(x,£)\right] \quad \theta>0, x\in R \dots (2)$$

£ : It is known as the feature vector of the base distribution.

$\bar{F}$: It is the survival function of the base distribution.

R: is the set of real numbers.

θ: Parameter of the exponential family.

$F(x,£)$: It is known as the cumulative (collective) function

(C.D.F) of the base distribution (Pareto distribution), and it is as follows:

$$F = (x, \alpha, k) = 1 - (\frac{k}{x})^{\alpha} \qquad x \geq k\ ;\ \alpha, k > 0 \qquad \ldots (3)$$

$f(x, £)$: It is known as the probability density function (P.D.F) of the basic distribution (Pareto distribution), and it is represented as follows:

$$\text{f}(\text{x}, \alpha, k) = \begin{cases} \frac{\alpha\, k^{\alpha}}{x^{\alpha+1}} & x \geq k\ \ ;\ k, \alpha > 0 \\ 0 & O.W \end{cases} \qquad \ldots (4)$$

By substituting equation (3) into equation (1) and simplifying, we get the cumulative distribution function for the (Pareto – Exponential) distribution as in equations (5) (6):

$$G_{\text{nlte-pa}}(\text{x}, \theta, \alpha, k) = 1 - \left(\frac{1 - F(x, \theta, \alpha, k)}{e^{\theta F(x, \theta, \alpha, k)}}\right)$$

$$;\ \theta, \alpha, k > 0\ ;\ x \geq k \quad \ldots (5)$$

$$G_{\text{nlte-pa}}(\text{x}, \theta, \alpha, k) = 1 - \frac{(\frac{k}{x})^{\alpha}}{e^{\theta(1-(\frac{k}{x})^{\alpha})}} \qquad ; x \geq k\ ; \theta, k, \alpha \quad > 0 \quad \ldots (6)$$

To obtain the probability density function for the (NLTE-Pareto) distribution, we substitute equation (3) and (4) into equation (2), whose steps are explained in equations (7) (8):

$$g_{\text{nlte-pa}}(x, \theta, \alpha, k) = \frac{f(x, \theta, \alpha, k)}{e^{\theta\, F(x, \theta, \alpha, k)}}\left[1 + \theta\big(1 - F(x, \theta, \alpha, k)\big)\right]$$

$$; \theta, \alpha, k > 0\ ;\ x \geq k \qquad \ldots (7)$$

$$g_{\text{nlte-pa}}\,(x_i,\theta,\alpha,k) \;=\; \frac{\frac{\alpha k^{\alpha}}{x^{\alpha+1}}}{e^{\theta\left(1-\left(\frac{k}{x}\right)^{\alpha}\right)}}\left[1+\theta(\frac{k}{x})^{\alpha}\right]$$

$$;x \geq k;\theta,k,\alpha > 0 \qquad \ldots(8)$$

We note that equation (8) represents the probability density function of the (Pareto-exponential) distribution, which fulfills the conditions of the probability density function, which states that its integration for all values of the random variable (x) is equal to one and its value is greater than zero, and figures (2), (1) reinforce that .

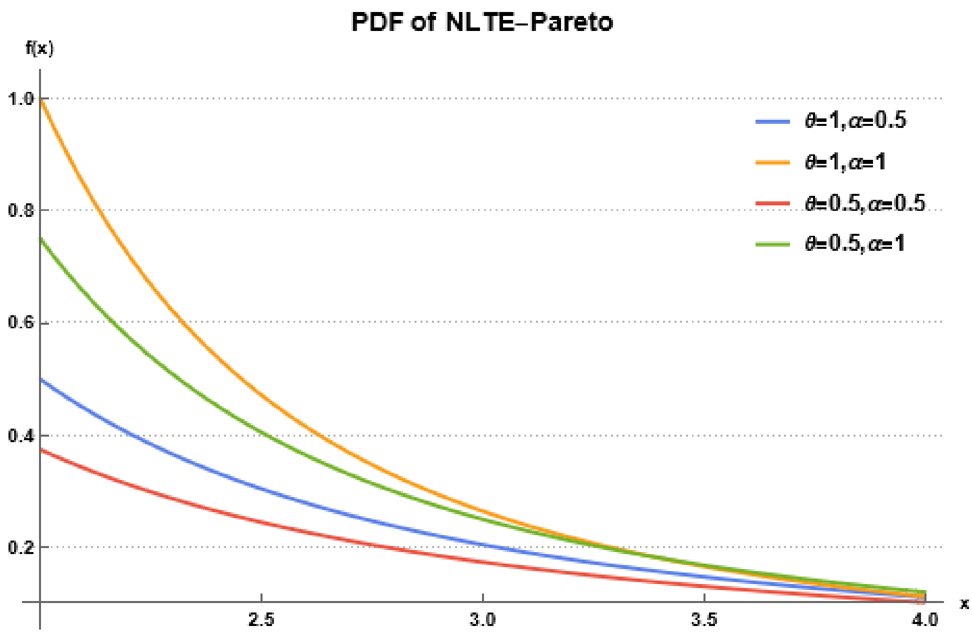

Figure (1): The probability density function of the (NLTE-Pa) distribution with different values of the parameters.

Source: Prepared by the researcher

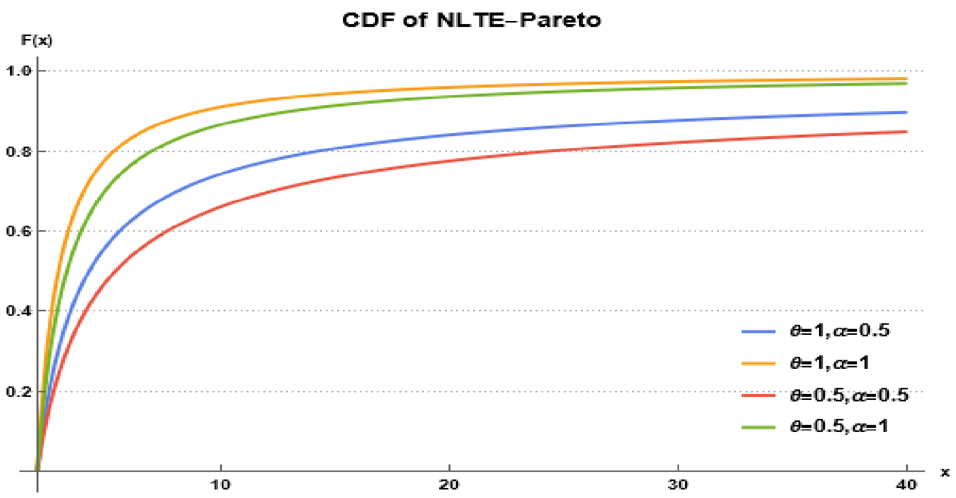

Figure (2) : Cumulative distribution function for (NLTE-Pa) distribution with different values of parameters.

Source: Prepared by the researcher

PROPERTIES

Moment Non-central

We assume that (X) is a random variable that follows the (NLTE-Pa) distribution, so the mathematical formula for the observed eccentric moment is known by the following formula:

$$\mu_r' = E(X^r) = \int_k^{\infty} x^r \; g_{nlte\_pa}(x_i, \alpha, k, \theta) \; dx \qquad \dots(9)$$

Substituting equation (8) into equation (9) and simplifying, we get:

$$\mu_r' = E(X^r) = \int_k^{\infty} x^r \frac{\frac{\alpha K^{\alpha}}{x^{\alpha+1}}}{e^{\theta\left(1-\left(\frac{k}{x}\right)^{\alpha}\right)}} \left[1 + \theta\left(\frac{k}{x}\right)^{\alpha}\right] \; dx$$

$$= \int_k^\infty x^{r-\alpha-1}\, \alpha\, k^\alpha\, e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)} \left[1+\theta(\frac{k}{x})^\alpha\right] dx$$

$$= \alpha\, k^\alpha \left[\int_k^\infty x^{r-\alpha-1}\; e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)}\; dx \right.$$

$$\left. + \theta\, k^\alpha \int_k^\infty x^{r-2\alpha-1}\; e^{-\theta(1-\left(\frac{k}{x}\right)^\alpha)}\; dx\right]$$

$$= \alpha\, k^\alpha [\mathrm{I} + \mathrm{II}] \qquad \ldots(10)$$

Simplifying the first term I:

$$= \int_k^\infty x^{r-\alpha-1}\, e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)}\quad dx$$

$$let \;\; y = \;\theta\left(1-(\frac{k}{x})^\alpha\right)$$

$$\frac{y}{\theta} = 1-\left(\frac{k}{x}\right)^\alpha \Rightarrow \frac{k^\alpha}{x^\alpha} = 1-\frac{y}{\theta} \Rightarrow x = \left(\frac{k^\alpha}{1-\frac{y}{\theta}}\right)^{\frac{1}{\alpha}}$$

$$x = k\left(1-\frac{y}{\theta}\right)^{-1/\alpha} \Rightarrow dx = \frac{k}{\alpha\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

For the purpose of knowing the limits of the variable (y_i), we substitute for the limits of the variable (x_i) according to the table below:

x	$y=\theta\left(1-\left(\frac{k}{x}\right)^{\alpha}\right)$
∞	θ
k	0

So the limits of the variable (y_i) are as follows: $0 < y < \theta$

$$\text{I} = \int_0^{\theta}\left(k\left(1-\frac{y}{\theta}\right)^{-1/\alpha}\right)^{r-\alpha-1} e^{-y}\ \frac{k}{\alpha\,\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

$$\text{I} = \frac{k^{r-\alpha}}{\alpha\,\theta}\int_0^{\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{r}{\alpha}} e^{-y}\,dy \qquad \dots(11)$$

Simplifying the second term II:

$$\text{II} = \theta\,k^{\alpha}\int_k^{\infty} x^{r-2\alpha-1}\ e^{-\theta\left(1-\left(\frac{k}{x}\right)^{\alpha}\right)}\ dx$$

By integrating in the same way as the first term, the result of the second term (II) is defined by equation (12):

$$\therefore\ \text{II} = \theta\,k^{\alpha}\int_0^{\theta}\left(k\left(1-\frac{y}{\theta}\right)^{-\frac{1}{\alpha}}\right)^{r-2\alpha-1} e^{-y}\ \frac{k}{\alpha\,\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

$$= \frac{k^{r-\alpha}}{\alpha}\int_0^{\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{r}{\alpha}+1} e^{-y}\ dy \qquad \dots(12)$$

By substituting equations (11) and (12) into equation (10), the observed Non-centric moment of the (NLTE-Pa) distribution is defined by equation (13):

$$\acute{\mu}_r = \alpha\, k^{\alpha} \left[\frac{k^{r-\alpha}}{\alpha\, \theta} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{r}{\alpha}} e^{-y}\, dy + \frac{k^{r-\alpha}}{\alpha} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{r}{\alpha}+1} e^{-y}\, dy \right] \quad \ldots(13)$$

And when (r = 1) we get the first Non-centric moment of the (NLTE-Pa) distribution, as in equation (14) below:

$$\boldsymbol{\acute{\mu}_1 = \alpha\, k^{\alpha} \left[\frac{k^{1-\alpha}}{\alpha\, \theta} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{1}{\alpha}} e^{-y}\, dy + \frac{k^{1-\alpha}}{\alpha} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{1}{\alpha}+1} e^{-y}\, dy \right] \quad \ldots(14)}$$

When (r=2) we get the second Non-centric moment of the (NLTE-Pa) distribution, defined by equation (15):

$$\boldsymbol{\acute{\mu}_2 = \alpha\, k^{\alpha} \left[\frac{k^{2-\alpha}}{\alpha\, \theta} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{2}{\alpha}} e^{-y}\, dy + \frac{k^{2-\alpha}}{\alpha} \int_0^{\theta} \left(1 - \frac{y}{\theta}\right)^{-\frac{2}{\alpha}+1} e^{-y}\, dy \right] \quad \ldots(15)}$$

Central Moments

We assume that (X) is a random variable that follows the (NLTE-Pa) distribution, so the mathematical formula for the central moment seen around the mean is defined by the following formula:

$$\mu_r = E(X-\mu)^r = \int_k^\infty (x-\mu)^r \, g_{\text{nlte-pa}}(x) \; dx \qquad \dots (16)$$

Substituting equation (8) into (16), we get:

$$= \int_k^\infty (x-\mu)^r \; \frac{\frac{\alpha\, k^\alpha}{x^{\alpha+1}}}{e^{\theta\left(1-(\frac{k}{x})^\alpha\right)}} \left[1+\theta(\frac{k}{x})^\alpha\right] \; dx \qquad \dots (17)$$

Taking advantage of the binomial theorem, it is possible to simplify equation (17) as follows:

$$\mu_r = \int_k^\infty \sum_{i=0}^{r} \binom{r}{i} x^i \, (-\mu)^{r-i} \; \frac{\frac{\alpha\, k^\alpha}{x^{\alpha+1}}}{e^{\theta\left(1-(\frac{k}{x})^\alpha\right)}} \left[1+\theta(\frac{k}{x})^\alpha\right] \; dx$$

$$= \sum_{i=0}^{r} \binom{r}{i} (-\mu)^{r-i} \, \alpha k^\alpha \int_k^\infty x^{i-\alpha-1} \; e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)} \left[1+\theta(\frac{k}{x})^\alpha\right] \; dx$$

$$= \sum_{i=0}^{r} \binom{r}{i} (-\mu)^{r-i} \, \alpha k^\alpha \; \left[\int_k^\infty x^{i-\alpha-1} \; e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)} \; dx \right.$$

$$\left. + \int_k^\infty x^{i-\alpha-1} \; \frac{\theta k^\alpha}{x^\alpha} \; e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)} \; dx \right]$$

$$= \sum_{i=0}^{r} \binom{r}{i} (-\mu)^{r-i} \, \alpha k^\alpha \, [Z_1 + Z_2] \qquad \dots (18)$$

Simplifying the first term Z_1:

$$Z_1 = \int_k^\infty x^{i-\alpha-1} \; e^{-\theta\left(1-(\frac{k}{x})^\alpha\right)} \; dx$$

In the same way as the eccentric moment, we find that:

$$let \quad y = \theta\left(1 - (\frac{k}{x})^{\alpha}\right)$$

$$\frac{y}{\theta} = 1 - \left(\frac{k}{x}\right)^{\alpha} \Rightarrow \frac{k^{\alpha}}{x^{\alpha}} = 1 - \frac{y}{\theta} \Rightarrow x = \left(\frac{k^{\alpha}}{1 - \frac{y}{\theta}}\right)^{\frac{1}{\alpha}}$$

$$x = k\left(1 - \frac{y}{\theta}\right)^{-1/\alpha} \Rightarrow dx = \frac{k}{\alpha\,\theta}\left(1 - \frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

For the purpose of knowing the limits of the variable (y_i), we substitute for the limits of the variable (x_i) according to the table below:

x	$y = \theta\left(1 - \left(\frac{k}{x}\right)^{\alpha}\right)$
∞	θ
k	0

So the limits of the variable (y_i) are as follows: $0 < y < \theta$ and therefore:

$$Z_1 = \int_0^{\theta}\left(k\left(1 - \frac{y}{\theta}\right)^{-\frac{1}{\alpha}}\right)^{i-\alpha-1} e^{-y}\,\frac{k}{\alpha\,\theta}\left(1 - \frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

$$Z_1 = \frac{k^{i-\alpha}}{\alpha\,\theta}\int_0^{\theta}\left(1 - \frac{y}{\theta}\right)^{-i/\alpha} e^{-y}\,dy \qquad \ldots(19)$$

Simplify the second term Z_2:

$$Z_2 = \int_k^{\infty} x^{i-\alpha-1} \frac{\theta\, k^{\alpha}}{x^{\alpha}} e^{-\theta\left(1-(\frac{k}{x})^{\alpha}\right)} dx$$

In the same way as the first term, the second term takes the following form:

$$Z_2 = \int_0^{\theta} \left(k\left(1-\frac{y}{\theta}\right)^{-1/\alpha}\right)^{i-\alpha-1} \frac{\theta\, k^{\alpha}\, e^{-y}}{\left(k\left(1-\frac{y}{\theta}\right)^{-1/\alpha}\right)^{\alpha}} \frac{k}{\alpha\theta}\left(1-\frac{y}{\theta}\right)^{-\frac{1}{\alpha}-1} dy$$

$$Z_2 = \frac{k^{i-\alpha}}{\alpha} \int_0^{\theta} \left(1-\frac{y}{\theta}\right)^{-\frac{i}{\alpha}+1} e^{-y}\, dy \qquad \dots(20)$$

$$\mu_r = \sum_{i=0}^{r} \binom{r}{i} (-\mu)^{r-i}\, \alpha k^{\alpha} \left[\frac{k^{i-\alpha}}{\alpha\,\theta} \int_0^{\theta} \left(1-\frac{y}{\theta}\right)^{-\frac{i}{\alpha}} e^{-y}\, dy + \frac{k^{i-\alpha}}{\alpha} \int_0^{\theta} (1-\frac{y}{\theta})^{-\frac{i}{\alpha}+1} e^{-y}\, dy\right] \qquad \dots(21)$$

In order to find the central moments in any order, it is possible to substitute the values of r into equation (21).

Coefficient of Skewness:

The mathematical formula for the torsion coefficient can be derived through the following law:

$$C.S = \frac{\mu_3}{\sqrt{(\mu_2)^3}}$$

We substitute the second centripetal moment and the third centripetal moment into the mathematical formula to get the Coefficient of Skewness.

Coefficient of Kurtosis:

The mathematical formula for the coefficient of flattening can be derived through the following law:

$$C.K = \frac{\mu_4}{(\mu_2)^2}$$

And by substituting the second central moment and the fourth central moment in the mathematical formula, we get the Coefficient of Kurtosis.

Coefficient of Variation

We substitute the second central moment(μ_2) and the first eccentric moment ($\acute{\mu}_1$) to obtain the value of the coefficient of variation, i.e.:

$$C.V = \frac{\sqrt[2]{\mu_2}}{\acute{\mu}_1} * 100$$

Median:

The mathematical formula for the median of the random variable x can be defined as follows:

$$G_{\text{nlte-pa}}(x) = 0.5 \qquad \ldots(22)$$

Substituting equation (6) into equation (22), we get:

$$1 - \frac{(\frac{k}{x})^{\alpha}}{e^{\theta\left(1-(\frac{k}{x})^{\alpha}\right)}} = 0.5$$

$$1 - 0.5 = \frac{(\frac{k}{x})^{\alpha}}{e^{\theta\left(1-(\frac{k}{x})^{\alpha}\right)}}$$

$$0.5 = \frac{(\frac{k}{x})^{\alpha}}{e^{\theta\left(1-(\frac{k}{x})^{\alpha}\right)}}$$

$$0.5e^{\theta\left(1-(\frac{k}{x})^{\alpha}\right)} = (\frac{k}{x})^{\alpha}$$

We take (ln) for both sides:

$$ln0.5 + \theta\left(1-(\frac{k}{x})^{\alpha}\right) = ln(\frac{k}{x})^{\alpha} \qquad \ldots (23)$$

We find that the non-linear equation cannot be solved by the usual analytical methods, and accordingly, numerical methods are resorted to, including the (Newton-Raphson) method to obtain the median (Med).

ESTIMATION

Maximum Likelihood Estimators(MLE)

This method is considered one of the important methods for estimating the parameters of probability distributions because it contains good characteristics, including the property of impartiality and is sufficient, and has the least possible variance, and non-covariance or stability, which means that the parameter estimated using this method contains the largest amount of information and when it is replaced in functions Others, such as the reliability function, these estimates remain as great as possible, and are more accurate than other estimation methods when the sample size is large. This method can be defined as the greatest possible estimator that makes the logarithm of the potential function at its end. The first to introduce this method is the scientist (C.F.Gauss) and the researcher (R.A.Fisher) applied it for the first time in 1920. **(AbdulHassan,2022:42),(Qasim,2021:16),(Abody,2016:404),(Sahib,2020:34)** We assume that (xi) is a random variable that follows the (NLTE-Pa) distribution, and then the maximum possibility function represents the common function of the independent random variables (x_1, x_2, x_3… x_n) and my agency:

$$\boldsymbol{L(x_1 \dots x_n\ ;\ \alpha,\theta,k) = g_{\text{nlte-pa}}\ (x_1;\alpha,\theta,k).\, g_{\text{nlte-pa}}(x_2;\alpha,\theta,k) \dots g_{\text{nlte-pa}}(x_n,\alpha,\theta,k)}$$

And equations (24) (25) below determine the maximum potential function for the (NLTE-Pa) distribution:

$$= \prod_{i=1}^{n} \frac{\frac{\alpha k^{\alpha}}{{x_i}^{\alpha+1}}}{e^{\theta\left(1-(\frac{k}{x_i})^{\alpha}\right)}} \left[1+\theta(\frac{k}{x_i})^{\alpha}\right] \qquad \dots(24)$$

$$lnL(x_1 \dots x_n\ ;\ \alpha,\theta,k) = n\, ln\alpha + n\alpha\, lnk - \mathrm{n}\theta + \theta \sum_{i=1}^{n} \left(\frac{k}{x_i}\right)^{\alpha}$$

$$-\alpha \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} lnx_i + \sum_{i=1}^{n} ln\left(1+\theta(\frac{k}{x_i})^{\alpha}\right) \qquad \dots(25)$$

We take the partial derivative of equation (25) and set it equal to zero for the parameters (α,θ).

$$\frac{\partial lnL}{\delta\alpha} = \frac{n}{\alpha} + nlnk + \theta\left[\sum_{i=1}^{n}\left(\frac{k}{x_i}\right)^{\alpha} lnk - \sum_{i=1}^{n}\left(\frac{k}{x_i}\right)^{\alpha} lnx_i\right] - \sum_{i=1}^{n} lnx_i$$

$$+ \sum_{i=1}^{n} \frac{\theta\left[\sum_{i=1}^{n}\left(\frac{k}{x_i}\right)^{\alpha} lnk - \sum_{i=1}^{n}\left(\frac{k}{x_i}\right)^{\alpha} lnx_i\right]}{\left(1+\theta(\frac{k}{x_i})^{\alpha}\right)}$$

$$=0 \quad \ldots (26)$$

$$\frac{\partial lnL}{\delta\theta} = -n + \left(\frac{k}{\sum_{i=1}^{n} x_i}\right)^{\alpha} + \sum_{i=1}^{n} \frac{(\frac{k}{x_i})^{\alpha}}{\left(1+\theta(\frac{k}{x_i})^{\alpha}\right)}$$

$$\frac{\partial lnL}{\delta\theta} = \left(\frac{k}{\sum_{i=1}^{n} x_i}\right)^{\alpha} + \sum_{i=1}^{n} \frac{(\frac{k}{x_i})^{\alpha}}{\left(1+\theta(\frac{k}{x_i})^{\alpha}\right)} - n$$

$$=0 \quad \ldots (27)$$

As we note that equations (26), (27) are non-linear equations and therefore cannot be solved by the usual analytical methods, and therefore we will resort to solving them by numerical methods, including the (Newton Raphson) method to obtain the greatest possible estimations for the unknown parameters $(\hat{\alpha}_{mle}, \hat{\theta}_{mle})$.

Using arranged statistics, we find that:

$$\hat{k} = \min(x_i) \quad \ldots (28)$$

Least Square Method (LS)

This method is based on finding the parameters that make the sum of the squares of the error as low as possible and is characterized by its unbiased nature and consistency of the estimators and can be formulated as follows**:(Abdul-Hassan,2022: 44),(Sharhan and Rahi,2019: 502)**

$$\dots (29) \boldsymbol{S} = \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \left(G_{\text{nlte-pa}}(\boldsymbol{x_i}) - \frac{\boldsymbol{i}}{\boldsymbol{n}+\mathbf{1}} \right)^{\mathbf{2}}$$

Since $\frac{\boldsymbol{i}}{\boldsymbol{n}+\mathbf{1}}$: represents the nonparametric estimate of the cumulative distribution function.

By performing the partial derivation of equation (29) with respect to the parameters of the distribution, we obtain (α, θ) equations (30), (31) after equating them to zero, as follows:

$$\frac{\partial S}{\partial \alpha} = 2 \sum_{i=1}^{n} \left(G_{nlte-pa}(x_i) - \frac{i}{n+1} \right) . \left[\frac{\partial G_{nlte-pa}(x_i; \alpha, \theta)}{\partial \alpha} \right] = 0 \qquad \dots (30)$$

$$\frac{\partial S}{\partial \theta} = 2 \sum_{i=1}^{n} \left(G_{nlte-pa}(x_i) - \frac{i}{n+1} \right) . \left[\frac{\partial G_{nlte-pa}(x_i; \alpha, \theta)}{\partial \theta} \right] = 0 \qquad \dots (31)$$

We extract the derivatives $\left(\left[\frac{\partial G_{\boldsymbol{nlte-pa}}(x_i; \alpha, \theta)}{\partial \theta} \right], \left[\frac{\partial G_{\boldsymbol{nlte-pa}}(x_i; \alpha, \theta)}{\partial \alpha} \right] \right)$.

$$\frac{\partial G_{nlte-pa}(\boldsymbol{x_i}; \boldsymbol{\alpha}, \boldsymbol{\theta})}{\partial \boldsymbol{\alpha}} = - \boldsymbol{e}^{-\boldsymbol{\theta}\left(\mathbf{1}-\left(\frac{\boldsymbol{k}}{\boldsymbol{x_i}}\right)^{\boldsymbol{\alpha}}\right)} \boldsymbol{\theta}\left(\frac{\boldsymbol{k}}{\boldsymbol{x_i}}\right)^{\mathbf{2}\boldsymbol{\alpha}} \boldsymbol{ln}\left(\frac{\boldsymbol{k}}{\boldsymbol{x_i}}\right) - \boldsymbol{e}^{-\boldsymbol{\theta}\left(\mathbf{1}-\left(\frac{\boldsymbol{k}}{\boldsymbol{x_i}}\right)^{\boldsymbol{\alpha}}\right)} \left(\frac{\boldsymbol{k}}{\boldsymbol{x_i}}\right)^{\boldsymbol{\alpha}} \boldsymbol{lnk} \qquad \dots (32)$$

$$\frac{\partial G_{nlte-pa}(x_i; \alpha, \theta)}{\partial \theta} = -\left(\frac{k}{x_i}\right)^{\alpha} e^{-\theta\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right)} \left(-\left(1 - \left(\frac{k}{x_i}\right)^{\alpha} \right) \right)$$

$$\frac{\partial G_{nlte-pa}\,(x_i;\ \alpha,\theta)}{\partial\theta} = \left(\frac{k}{x_i}\right)\left(1-(\frac{k}{x_i})^{\alpha}\right)\ e^{-\theta\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right)} \quad \dots (33)$$

After substituting equations (33), (32) and the cumulative distribution function for the distribution of (NLTE-Pa) in equation (31), (30), he obtains non-linear equations and therefore cannot be solved by the usual analytical methods, and therefore we will resort to solving them by numerical methods, including the method (Newton-Raphson) to obtain least squares estimates for the unknown parameters $(\hat{\alpha}_{ls}, \hat{\theta}_{ls})$.

As for the parameter k, its estimate is as in equation (28).

Methods of Anderson-Darling Estimation (ADEs)

The Anderson-Darling estimates (ADEs) of the (α,θ,k) parameters of the (NLTE-Pa) distribution can be determined by reducing the Anderson-Darling function for the parameters (θ,α,k), and thus the Anderson-Darling can be obtained by the following equation**:(ZeinEldin et al.,2019: 16)**

$$A(\theta,\alpha,k) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right] - H_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right\} \quad \dots (34)$$

Since:

$\boldsymbol{H_{nlte-pa}}\left(x_{(n+1-i)};\alpha,\theta,k\right)$: It represents the cumulative hazard rate function of the distribution (NLTE-Pa).

$\boldsymbol{G_{nlte-pa}}\left(x_{(i)};\alpha,\theta,k\right)$: It represents the cumulative distribution function of the distribution (NLTE-Pa).

Derivation of equation (34) is a partial derivation with respect to the parameters (θ, α), then we equate it to zero, to get after that:

$$\frac{\partial A(\theta,\alpha,k)}{\partial\theta}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)/\partial\theta}{G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)}-\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)/\partial\theta}{1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)}\right]=0 \quad \ldots (35)$$

$$\frac{\partial A(\theta,\alpha,k)}{\partial\alpha}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)/\partial\alpha}{G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)}-\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)/\partial\alpha}{1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)}\right]=0 \quad \ldots (36)$$

We extract the derivatives with respect to the parameter (θ) $\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)}{\partial\theta}$ $\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)}{\partial\theta}$.

$$\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)}{\partial\theta}=\left(\frac{k}{x_i}\right)^{\alpha}e^{-\theta\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right)}\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right) \quad \ldots (37)$$

$$\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)}{\partial\theta} = -\left(\frac{k}{x_{n+1-i}}\right)^{\alpha} e^{-\theta\left(1-\left(\frac{k}{x_{n+1-i}}\right)^{\alpha}\right)}\left(1-\left(\frac{k}{x_{n+1-i}}\right)^{\alpha}\right) \quad \ldots(38)$$

We extract the derivatives with respect to the parameter (α) $\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)}{\partial\theta}$ $\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)}{\partial\theta}$.

$$\frac{\partial\left(G_{nlte-pa}\left(x_{(i)};\alpha,\theta,k\right)\right)}{\partial\alpha} = -e^{-\theta\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right)}\theta\left(\frac{k}{x_i}\right)^{2\alpha} ln\frac{k}{x_i} - e^{-\theta\left(1-\left(\frac{k}{x_i}\right)^{\alpha}\right)}\left(\frac{k}{x_i}\right)^{\alpha} ln\frac{k}{x_i} \quad \ldots(39)$$

$$\frac{\partial\left(1-G_{nlte-pa}\left(x_{(n+1-i)};\alpha,\theta,k\right)\right)}{\partial\alpha} = e^{-\theta\left(1-\left(\frac{k}{x_{n+1-i}}\right)^{\alpha}\right)}\theta\left(\frac{k}{x_{n+1-i}}\right)^{2\alpha} ln\frac{k}{x_{n+1-i}} + e^{-\theta\left(1-\left(\frac{k}{x_{n+1-i}}\right)^{\alpha}\right)}\left(\frac{k}{x_{n+1-i}}\right)^{\alpha} ln\frac{k}{x_{n+1-i}} \quad \ldots(40)$$

We substitute the cumulative distribution function of the (Pareto-exponential) distribution and equations (38), (37) into equation (35) and equations (40), (39) into equation (36) and after substituting in equations (35) (36), we notice that the equations It is non-linear and therefore cannot be solved by the usual analytical methods. Therefore, we will resort to solving it by numerical methods, including the (Newton-Raphson) method to obtain Anderson-Darling estimations for the unknown parameters $(\hat{\alpha}_{ls}, \hat{\theta}_{ls})$. As for the parameter (k), it is estimated according to equation (28).

SIMULATION

In order to make a comparison between the estimation methods (the greatest possibility method, the method of least squares, and the Anderson Darling method) using the mean square error (MSE) criterion, the simulation experiment was carried out based on five sample sizes (25, 50, 75, 100, and 125) and six models of default values. For distribution parameters (NLTE-Pa), the experiment was repeated 1000 times, and below are the simulation results for each model .

Table (1): It represents the estimated parameter values and the mean squared error at the default values (θ = 1, α = 0.5, k = 1.5)

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	0.9023	0.941	0.833	0.0305	0.9111	0.8889	MLE
	α	0.546	0.5866	0.6289	0.0251	0.0961	0.0883	MLE
	k	1.5547	1.4661	1.5687	0.0086	0.0116	0.0091	MLE
50	θ	0.9536	1.0667	1.1263	0.0139	0.9304	1.2956	MLE
	α	0.4989	0.5474	0.5519	0.0138	0.0479	0.0499	MLE
	k	1.5165	1.4842	1.5325	0.0016	0.004	0.002	MLE
75	θ	0.9796	0.81305	1.0492	0.0098	0.6878	1.3682	MLE
	α	0.4856	0.6087	0.5755	0.0103	0.0615	0.0535	MLE
	k	1.5031	1.485	1.5162	0.0122	0.0031	0.0005	ADEs
100	θ	0.9863	0.9837	1.1215	0.0107	0.5455	1.0418	MLE
	α	0.47999	0.5628	0.5534	0.0122	0.0464	0.0479	MLE
	k	1.4955	1.4863	1.5137	0.0006	0.0025	0.0004	ADEs
125	θ	1.0026	1.0738	1.2248	0.0353	0.7225	1.2134	MLE
	α	0.4669	0.5381	0.5257	0.0103	0.0345	0.034	MLE
	k	1.4986	1.4856	1.5132	0.0005	0.0018	0.0003	ADEs

Table (1) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75,100,125) and the first model for the default values of the parameters ($\theta = 1$, $\alpha = 0.5$, $k = 1.5$).

At size n = 25, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0305, MSE = 0.0251, MSE = 0.0086, respectively.

At size n = 50, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0139, MSE = 0.0138, MSE = 0.0016

At size n = 75, the (MLE) method appeared to be the best method for parameter estimation (θ, α) with mean square error MSE = 0.0098 and MSE = 0.0103 respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of 0.0005 MSE = 0.0005.

At size n = 100, it was found that the (MLE) method is the best method for estimating the parameters (θ, α)) with mean square error MSE = 0.0107 and MSE = 0.0122, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0.0004 .

At size n=125, it was obtained that the (MLE) method is the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0353, MSE = 0.0103, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0. 0003 .

Table (2): It represents the estimated parameter values and average of error boxes for default values (θ = 1, α = 0.5, k = 2).

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	0.7215	1.0106	1.0417	0.1104	0.8015	0.8813	MLE
	α	0.5877	0.5612	0.5754	0.0287	0.0716	0.0673	MLE
	k	2.0554	1.9688	2.0789	0.0117	0.0299	0.0149	MLE
50	θ	0.7715	0.9373	1.1375	0.0814	0.9067	1.4701	MLE
	α	0.5752	0.5868	0.5633	0.0138	0.0592	0.0510	MLE
	k	2.0331	1.9688	2.0456	0.0041	0.0090	0.0046	MLE
75	θ	0.7878	1.0553	1.0357	0.0715	0.7664	0.7971	MLE
	α	0.5639	0.5425	0.5487	0.0104	0.0451	0.0397	MLE
	k	2.0181	1.9809	2.0306	0.0021	0.0052	0.0022	MLE
100	θ	0.7868	0.9357	1.1052	0.0544	0.6067	0.984	MLE
	α	0.5436	0.5665	0.5398	0.0052	0.0499	0.0407	MLE
	k	2.0084	1.9738	2.0213	0.0008	0.0038	0.0009	MLE
125	θ	0.8478	0.9043	1.0872	0.3481	0.6749	0.9128	MLE
	α	0.5333	0.5711	0.5368	0.0063	0.0441	0.0370	MLE
	k	2.0018	1.9815	2.0136	0.0005	0.0036	0.0004	ADEs

Table (2) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75,100,125) and the first model for the default values of the parameters (θ = 1, α = 0.5, k = 2).

At size n = 25, the (MLE) method appeared to be the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.1104, MSE = 0.0287, MSE = 0.0117, respectively.

At size n = 50, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0814 , MSE = 0.0138 , MSE = 0.0041, respectively.

At size n = 75, it was found that the (MLE) method is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0715 , MSE = 0.0104 , MSE = 0.0021, respectively.

At size n = 100, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0544 , MSE = 0.0052 , MSE = 0.0008 , respectively.

At size n = 125, the (MLE) method appeared to be the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0063 and MSE = 0.3481, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0.0004.

Table (3): It represents the estimated parameter values and mean squares error at default values (θ = 0.5, α = 1, k = 1.5).

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	1.0279	0.8184	0.7333	0.3683	0.8952	1.1158	MLE
	α	0.8248	0.9514	1.0284	0.0767	0.2002	0.1667	MLE
	k	1.5418	1.4837	1.5451	0.0037	0.0048	0.0039	MLE
50	θ	1.0167	1.0173	0.7416	0.36802	1.5662	0.9161	MLE
	α	0.8059	0.8915	0.9827	0.0791	0.1318	0.1007	MLE
	k	1.5166	1.4936	1.5215	0.0007	0.0017	0.0008	MLE
75	θ	0.9907	0.6058	0.6641	0.3024	0.6083	0.9120	MLE
	α	0.8096	1.0325	1.0257	0.0564	0.11	0.0955	MLE
	k	1.5072	1.4935	1.5108	0.0001	0.0013	0.0002	MLE
100	θ	0.9598	0.8613	0.7376	0.2588	0.8645	0.6641	MLE
	α	0.8142	0.9396	0.9921	0.0493	0.1183	0.1067	MLE
	k	1.5056	1.494	1.5091	0.0001	0.001	0.0002	MLE
125	θ	0.9893	0.7465	0.6707	0.2919	0.5552	0.4941	MLE
	α	0.8036	0.9537	0.9922	0.0468	0.0767	0.0632	MLE
	k	1.5045	1.4935	1.5088	0.0001	0.0007	0.0002	MLE

Table (3) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75, 100, 125) and the first model for the default values of the parameters (θ = 0.5, α = 1, k = 1.5).

At size n = 25, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.3683 , MSE = 0.0767 , MSE = 0.0037, respectively.

At size n = 50, it was found that the (MLE) method is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.3680 , MSE = 0.0791 , MSE = 0.0007, respectively.

At size n = 75, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.3024 , MSE = 0.0564 , MSE = 0.0001, respectively.

At size n = 100, the (MLE) method appeared to be the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.2588, MSE = 0.0493, MSE = 0.0001, respectively.

At size n = 125, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.2919 , MSE = 0.0468 , MSE = 0.0001, respectively.

Table (4) It represents the estimated parameter values and mean squares error at default values (θ = 0.1, α = 3, k = 1.5).

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	0.1063	0.5970	0.4817	0.0003	0.7004	0.6802	MLE
	α	3.0493	2.6555	2.9037	0.01946	0.5983	0.6924	MLE
	k	1.5186	1.4995	1.5166	0.0005	0.0006	0.0004	ADEs
50	θ	0.1004	0.5426	0.1951	0.0002	1.1649	0.2241	MLE
	α	3.0767	2.6153	2.9216	0.0175	0.8319	0.2613	MLE
	k	1.5116	1.5005	1.5077	0.0002	0.0002	0.0001	ADEs
75	θ	0.1426	0.4651	0.3743	0.0002	0.5371	0.4157	MLE
	α	2.9885	2.5165	2.6425	0.0119	0.63391	0.4862	MLE
	k	1.5034	1.4985	1.5025	0.0002	0.0001	0.00001	ADEs
100	θ	0.0968	0.3309	0.2059	0.0002	0.3023	0.1614	MLE
	α	3.0437	2.661	2.7752	0.0106	0.42253	0.2755	MLE
	k	1.51	1.5062	1.5071	0.0001	0.0001	0.0002	MLE
125	θ	0.1033	0.5381	0.4275	0.00001	1.0186	0.7357	MLE
	α	3.0025	2.5553	2.6515	0.0007	0.6382	0.5139	MLE
	k	1.5044	1.5024	1.5034	0.00002	0.0003	0.0001	MLE

Table (4) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75, 100, 125) and the first model for the default values of the parameters (θ = 0.1, α = 3, k = 1.5).

At size n = 25, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α) with mean square error MSE = 0.0003 and MSE = 0.01946, respectively, while the

(ADEs) method was the best for estimating parameter (k) with an average of MSE = 0.0004

At size n = 50, it was found that the (MLE) method is the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0002 and MSE = 0.0175, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE = 0.0001.

At size n = 75, the (MLE) method appeared to be the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0002 and MSE = 0.0119 respectively, while the

(ADEs) method was the best for estimating parameter (k) with mean squares MSE = 0.00002 .

At size n = 100, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0002, MSE = 0.0106 and MSE = 0.0001, respectively.

At size n = 125, the (MLE) method appeared to be the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.00001 , MSE = 0.0007 , MSE = 0.00002, respectively.

Table (5) It represents the estimated parameter values and mean squares error at the default values (θ = 0.1, α = 3, k = 2).

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	0.18	0.7325	0.8117	0.0385	0.7461	0.9687	MLE
	α	3.0053	2.2069	2.3882	0.0895	1.2535	1.2546	MLE
	k	2.021	1.9706	2.0201	0.0009	0.0029	0.0008	ADEs
50	θ	0.1033	0.5439	0.4604	0.0074	0.9845	0.5288	MLE
	α	3.0369	2.4566	2.6899	0.0308	1.2814	0.5933	MLE
	k	2.0101	2.0033	2.0201	0.0001	0.0008	0.0002	MLE
75	θ	0.1060	0.4395	0.3605	0.0041	0.3766	0.29	MLE
	α	3.0047	2.603	2.7009	0.0263	0.6824	0.4359	MLE
	k	2.0082	1.9983	2.0074	0.0002	0.0003	0.0001	ADEs
100	θ	0.0981	0.3501	0.3280	0.0004	0.1836	0.1742	MLE
	α	3.0530	2.5266	2.6184	0.0248	0.4143	0.351	MLE
	k	2.0059	1.9955	2.0046	0.0001	0.0002	0.00004	ADEs
125	θ	0.1153	0.4593	0.2940	0.0002	0.5184	0.2300	MLE
	α	2.9448	2.6094	2.7404	0.0209	0.6123	0.2851	MLE
	k	2.0067	2.0039	2.0045	0.0001	0.0002	0.00004	ADEs

Table (5) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75, 100, 125) and the first model for the default values of the parameters (θ = 0.1, α = 3, k = 2) .

At size n = 25, it was obtained that the (MLE) method is the best method for estimating the parameters (θ, α, k) with mean squares error MSE = 0.0385 , MSE = 0.0895, respectively, while the

(ADEs) method was the best for estimating parameter (k) with an average of 0.0008 MSE = 0.0008 .

At size n = 50, the (MLE) method was obtained, which is the best method for estimating all parameters (θ, α, k) with mean square error MSE = 0.0074 , MSE = 0.0308 , MSE = 0.0001, respectively.

At size n = 75, the (MLE) method appeared to be the best method for parameter estimation (θ, α) with mean square error MSE = 0.0041, MSE = 0.0263 respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0.0001.

At size n = 100, it was found that the (MLE) method is the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0004 and MSE = 0.0248, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0.00004 .

At size n=125, the (MLE) method appeared to be the best method for estimating the parameters (θ, α)) with mean square error MSE = 0.0002 and MSE = 0.0209, respectively, while the (ADEs) method was the best for estimating parameter (k) with mean square error MSE = 0.00004.

Table (6) It represents the estimated parameter values and mean squares error at default values (θ = 0.1, α = 3, k = 2.5).

sample size	parameters	parameters			MSE			best
		MLE	LS	ADEs	MLE	LS	ADEs	
25	θ	.11310	0.7760	0.8038	0.0003	1.0596	1.2162	MLE
	α	3.0094	2.4830	2.5417	0.0003	1.2559	0.9055	MLE
	k	2.5243	2.4830	2.5222	0.0014	0.0022	0.0013	ADEs
50	θ	0.1122	0.8144	0.6824	0.0002	0.9068	0.6855	MLE
	α	3.0076	2.1138	2.3473	0.0002	1.0976	0.7551	MLE
	k	2.5195	2.4809	2.5179	0.0005	0.0012	0.0004	ADEs
75	θ	0.1126	0.3942	0.4703	0.0002	0.4760	0.3898	MLE
	α	2.9845	2.5519	2.5150	0.0001	0.6756	0.7092	MLE
	k	2.5081	2.4838	2.506	0.0001	0.0007	0.00001	ADEs
100	θ	0.1101	0.7329	0.2704	0.0001	1.1680	0.2775	MLE
	α	3.0075	2.4003	2.829	0.0001	1.0758	0.4127	MLE
	k	2.5146	2.5086	2.5131	0.00003	0.0003	0.0002	MLE
125	θ	0.1132	0.1884	0.1056	0.0001	00384	0.0143	MLE
	α	3.0028	2.9447	3.0781	0.00006	0.2972	0.2806	MLE
	k	2.5082	2.5035	2.5058	0.00003	0.0003	0.0001	MLE

Table (6) shows the results of the comparison between the estimation methods for the (NLTE-Pa) distribution with different sample sizes (25, 50, 75, 100, 125) and the first model for the default values of the parameters (θ = 0.1, α = 3, k = 2.5).

At size n = 25, it was obtained that the (MLE) method is the best method for estimating the parameters (θ, α, k) with mean square error MSE = 0.0003 , MSE = 0.0003 respectively, while the

(ADEs) method was the best for estimating parameter (k) with an average of MSE = 0.0013 .

At size n = 50, the (MLE) method appeared to be the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0002 and MSE = 0.0002, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of MSE =0.0004 .

At size n = 75, it was found that the (MLE) method is the best method for estimating the parameters (θ, α) with mean square error MSE = 0.0001 and MSE = 0.0002, respectively, while the (ADEs) method was the best for estimating parameter (k) with mean square error MSE = 0.00001.

At size n = 100, it was obtained that the (MLE) method is the best method for estimating the parameters (θ, α, k) with mean square error MSE = 0.0001, MSE = 0.0001, respectively, while the (ADEs) method was the best for estimating parameter (k) with an average of 0 MSE = 0.00003 .

At size n=125, the (MLE) method appeared to be the best method for estimating all parameters (θ, α, k) with mean square error MSE=0.0001, MSE=0.00006, MSE=0.00003, respectively.

CONCLUSIONS

The following points summarize the most important conclusions reached by the research.

1- Building a probabilistic model as a member of the family of life time distributions, which is a development of the Pareto distribution of the first type.

2- The Pareto exponential distribution is better than the exponential distribution because it has a non-stationary risk function .

3- The advantage of the maximum likelihood method for estimating the parameters of the Pareto-exponential distribution over the estimation methods specified in the research because it has the least mean square error.

4- The rest of the estimation methods come in preference after the maximum possibility method, which are respectively the Anderson-Darling method (ADEs) and the least squares method (OL).

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